UNIQUE COMMON COUPLED FIXED POINT THEOREM FOR FOUR MAPS SATISFYING GENERALIZED α -WEAKLY CONTRACTIVE CONDITIONIN PARTIAL METRIC SPACES

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Abstract

In this paper we obtain a unique common coupled fixed point theorem for four maps satisfying generalized α - weakly contractive condition and we give an example to illustrate our main theorem. Our result generalize and improve the theorem of Seonghoon Cho [26].

Keywords: Partial metric space, w - compatible maps, α - admissible function and Coupled fixed point.

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1. Introduction

There are many generalizations of the concept of metric spaces in the literature. One of them is a partial metric space introduced by Matthews [18] as a part of study of denotational semantics of data flow networks. After that fixed and common fixed point results in partial metric spaces were studied by many other authors, for example [2, 4, 5, 8, 9, 20].Throughout this paper, R^+ and N denote the set of all non-negative real numbers and the set of all positive integers respectively.
First we recall some basic definitions and lemmas which play crucial role in the theory of partial metric spaces.

Definition 1.1 *([14]):* A partial metric on a non empty set X is a function $p: X \times X \to R^+$ such that for all $x, y, z \in X$,

(p_1) $x = y \Leftrightarrow p(x, x) = p(x, y) = p(y, y)$, ($p_2$) $p(x, x) \le p(x, y), p(y, y) \le p(x, y)$,

(p_3) $p(x, y) = p(y, x)$, (p_4) $p(x, y) \le p(x, z) + p(z, y) - p(z, z)$.

The pair (X, p) is called a partial metric space (PMS).

If p is a partial metric on X, then the function $d_p: X \times X \to R^+$ given by

$d_p(x, y) = 2p(x, y) - p(x, x) - p(y, y)$ is a metric on X. It is clear that (i) $p(x, y) = 0 \Rightarrow x = y$,

(ii) $x \ne y \Rightarrow p(x, y) > 0$ and (iii) p(x, x) may not be 0.

Example 1.2 *(see e.g. [2, 8, 14]) :* Consider $X = R^+$ with $p(x, y) = \max\{x, y\}$. Then (X, p) is a partial metric space. It is clear that p is not a (usual) metric. Note that in this case $d_p(x, y) = |x - y|$.
We now state some basic topological notations (such as convergence, completeness, continuity) on partial metric spaces (see e.g. [2, 5, 8, 9,14]).
Definition 1.3: Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to be convergent to $x \in X$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n)$.

(ii) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if $\lim_{n, m\to\infty} p(x_n, x_m)$ exists and is finite.

(iii) (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p, to a point $x\in X$ such that $p(x, x) = \lim_{n, m\to\infty} p(x_n, x_m)$.

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The following lemma is one of the basic results in PMS ([2, 5, 8, 9,14]).

Lemma 1.4: Let (X, p) be a partial metric space.

(i) A sequence $\{x_n\}$ in (X, p) is said to be Cauchy sequence if and only if it is a Cauchy sequence in the metric space (X, d_p).

(ii) (X, p) is complete if and only if the metric space (X, d_p) is complete. Moreover $\lim_{n\to\infty} p(x, x_n) = 0$ if and only if $p(x, x) = \lim_{n\to\infty} p(x, x_n) = \lim_{n, m\to\infty} p(x_n, x_m)$.

Next we give a simple lemma which will be used in the proof of our main result. For the proof we refer to [2].

Lemma 1.5*([2]):* If $\{x_n\}$ converges to z in a partial metric space (X, p) and p(z, z) = 0 then $\lim_{n\to\infty} p(x_n, y) = p(z, y)$ for all $y \in X$.

Bhaskar and Lakshmikantham [7] introduced the the concept of coupled fixed points and lakshmikantham and Ciric [12] defined the common coupled fixed points. Later several authors obtained coupled fixed point theorems in various spaces, for example [1, 11, 18, 19, 21, 22, 24] and the references their in.

Definition 1.6 *([18]):* Let X be a non empty set, $F, G : X \times X \to X$, $f, g : X \to X$ be mappings. A point

$(x, y) \in X \times X$ is called a common coupled fixed point of F, G, f and g if F(x, y) = G(x, y) = fx = gx = x and

F(y, x) = G(y, x) = fy = gy = y.

Definition 1.7 ([1]): The mappings $S : X \times X \to X$ and $f : X \to X$ are called w-compatible if f(S(x, y)) = S(fx, fy) whenever f(x) = S(x, y) and f(y) = S(y, x).

Kaushik et al.[10] introduced by the following concept which is a generalization of the concept introduced by Mursaleen et al.[16].

Definition 1.8 *([10]) :* Let X be a non empty set and $\alpha : X^2 \times X^2 \to R^+$ be a function. Let $F: X \times X \to X$, $S : X \to X$ be mappings. Then F and S are said to be α - admissible if $\alpha((Sx, Sy), (Su, Sv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (F(u, v), F(v, u))) \geq 1$ for all $x, y, u, v \in X$.

If S = I (Identity map), then the above definition is the concept of Mursaleen et al.[16].

Recently Rao et al.[23] further extended the Definition 1.8 for four maps as follows:

Definition 1.9 *([23]):* Let X be a non empty set and $\alpha : X^2 \times X^2 \to R^+$ be a function. Let $F, G : X \times X \to X$ and $S, T : X \to X$ be mappings. Then we say that the pair (S, T) is α - admissible with respect to the pair (F, G) if

(i) $\alpha((Sx, Sy), (Tu, Tv)) \geq 1 \Rightarrow \alpha((F(x, y), F(y, x)), (G(u, v), G(v, u))) \geq 1$,

(ii) $\alpha((Tx, Ty), (Su, Sv)) \geq 1 \Rightarrow \alpha((G(x, y), G(y, x)), (F(u, v), F(v, u))) \geq 1$, for all $x, y, u, v \in X$.

We say that the pair (S, T) is triangular α-admissible with respect to the pair (F, G) if the pair (S, T) is α - admissible with respect to the pair (F, G) and $\alpha((x_1, y_1), (x_2, y_2)) \geq 1$, $\alpha((x_2, y_2), (x_3, y_3)) \geq 1 \Rightarrow \alpha((x_1, y_1), (x_3, y_3)) \geq 1$ for all $x_1, x_2, x_3, y_1, y_2, y_3 \in X$.

Note: $\max\{a + b, c + d\} \leq \max\{a, c\} + \max\{b, d\}$ for all $a, b, c, d \in R^+$.

In 1977, Alber et al. [3] generalized the Banach contraction principle by introducing the concept weak contraction mappings in Hilbert space and proved that every weak contraction mapping on a Hilbert space has a unique fixed point.

Rhodes [25] extended weak contraction principle in Hilbert spaces to metric spaces. Later many authors, for example,[6, 13, 15, 16, 17] obtained generalizations and extensions of the weak contraction principle to obtain fixed, common fixed, coupled and common coupled fixed point theorems in various spaces.

Definition 1.10: Let X be a non – empty set and $f : X \to R^+$. Then f is called lower semi continuous at $x \in X$ if $f(x) \le \lim_{n\to\infty} \inf f(x_n)$ whenever $\{x_n\} \subset X$ with $\lim_{n\to\infty} x_n = x$.

Let $\Psi = \{\psi : R^+ \to R^+$ such that ψ is continuous and $\psi(t) = 0 \Leftrightarrow t = 0\}$,

$\Phi = \{\phi : R^+ \to R^+$ such that ϕ is lower semi continuous and $\phi(t) = 0 \Leftrightarrow t = 0\}$.

Recently Seonghoon Cho [26] proved the following theorem.

Theorem 1.11*(Theorem 2.1 of [26]):* Let (X, d) be a complete metric space and $T : X \to X$ be a mapping satisfying $\psi(d(Tx, Ty) + \varphi(Tx) + \varphi(Ty) \le \psi(m(x, y, d, T, \varphi)) - \phi(l(x, y, d, T, \varphi))$ for all $x, y \in X$, where $\psi\in\Psi$, $\phi\in\Phi$, $\varphi : X \to R^+$ is a lower semi continuous function and

$$m(x, y, d, T, \varphi) = \max\left\{\begin{array}{l} d(x, y) + \varphi(x) + \varphi(y),\ d(x, Tx) + \varphi(x) + \varphi(Tx), \\ d(y, Ty) + \varphi(y) + \varphi(Ty), \\ \frac{1}{2}[d(x, Ty) + \varphi(x) + \varphi(Ty) + d(y, Tx) + \varphi(y) + \varphi(Tx)] \end{array}\right\}$$

and $l(x, y, d, T, \varphi) = \max\{d(x, y) + \varphi(x) + \varphi(y),\ d(y, Ty) + \varphi(y) + \varphi(Ty)\}$.

Then there exists a unique $z \in X$ such that $Tz = z$ and $\varphi(z) = 0$.

Using these concepts, we prove a unique common coupled fixed point theorem for four maps in partial metric spaces. Our theorem generalize and extend the Theorem 2.1 of Seonghoon Cho [26]. We also give an example to illustrate our theorem. We call the condition (2.1.3) as generalized α - weakly contractive condition associated with four maps involved in it.

Now we give our main result.

2. Main result

Now we prove a unique common coupled fixed point theorem without using orderedness in partial metric spaces.

Theorem 2.1: Let (X, p) be a partial metric space and $\alpha : X^2 \times X^2 \to R^+$ be a function. Let

$F, G: X \times X \to X$ and $S, T : X \to X$ be maps satisfying

(2.1.1) $F(X \times X) \subseteq T(X)$ and $G(X \times X) \subseteq S(X)$,

(2.1.2) either S(X) or T(X) is a complete sub space of X,

(2.1.3) $\alpha((Sx, Sy), (Tu, Tv))\,\psi[p(F(x, y), G(u, v)) + \varphi(F(x, y)) + \varphi(G(u, v))] \le \psi\left(M^{u,v}_{x,y}\right) - \phi\left(M^{u,v}_{x,y}\right)$

for all $x, y, u, v \in X$, $\psi\in\Psi$, $\phi\in\Phi$, $\varphi : X \to R^+$ is a lower semi continuous function and where

$$M^{u,v}_{x,y} = \max\left\{\begin{array}{l} p(Sx, Tu) + \varphi(Sx) + \varphi(Tu),\ p(Sy, Tv) + \varphi(Sy) + \varphi(Tv), \\ p(Sx, F(x, y)) + \varphi(Sx) + \varphi(F(x, y)),\ p(Sy, F(y, x)) + \varphi(Sy) + \varphi(F(y, x)), \\ p(Tu, G(u, v)) + \varphi(Tu) + \varphi(G(u, v)),\ p(Tv, G(v, u)) + \varphi(Tv) + \varphi(G(v, u)), \\ \frac{1}{2}[p(Sx, G(u, v)) + \varphi(Sx) + \varphi(G(u, v)) + d(Tu, F(x, y)) + \varphi(Tu) + \varphi(F(x, y))], \\ \frac{1}{2}[p(Sy, G(v, u)) + \varphi(Sy) + \varphi(G(v, u)) + d(Tv, F(y, x)) + \varphi(Tv) + \varphi(F(y, x))] \end{array}\right\}$$

(2.1.4) (a) $\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \ge 1$, (b) $\alpha((Sy_1, Sx_1), (F(y_1, x_1), F(x_1, y_1))) \ge 1$,

c) $\alpha((F(x_1, y_1), F(y_1, x_1)), (Sx_1, Sy_1)) \ge 1$ and (d) $\alpha((F(y_1, x_1), F(x_1, y_1)), (Sy_1, Sx_1)) \ge 1$ for some $x_1, y_1 \in X$.

(2.1.5) The pair (S, T) is triangular α-admissible with respect to the pair (F, G), (2.1.6) the pairs (F, S) and (G, T) are w - compatible,

(2.1.7) Assume that

(i) $\alpha((u, v), (z_{2n+1}, w_{2n+1})) \geq 1$, $\alpha((v, u), (w_{2n+1}, z_{2n+1})) \geq 1$ for all $n \in N$
(ii) $\alpha((Su, Sv), (z_{2n+1}, w_{2n+1})) \geq 1$, $\alpha((Sv, Su), (w_{2n+1}, z_{2n+1})) \geq 1$ for all $n \in N$
(iii) $\alpha((u, v), (u, v)) \geq 1$, $\alpha((v, u), (v, u)) \geq 1$.
(iv) $\alpha((u, v), (Tu, Tv)) \geq 1$, $\alpha((v, u), (Tv, Tu)) \geq 1$.

Whenever there exist sequences $\{z_n\}$ and $\{w_n\}$ in X such that $\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1$, $\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1$, $\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1$, $\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1$ with $z_n \to u$ and $w_n \to v$.

Then F, G, S and T have a common coupled fixed point.

(2.1.8) Further if we assume that $\alpha((u, v), (u^1, v^1)) \geq 1$, $\alpha((v, u), (v^1, u^1)) \geq 1$ whenever (u, v) and (u^1, v^1) are common coupled fixed points of F, G, S and T and then F, G, S and T have a unique common coupled fixed point in $X \times X$.

Proof : From (2.1.4), there exist x_1, y_1 in X such that

(a) $\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1$, (b) $\alpha((Sy_1, Sx_1), (F(y_1, x_1), F(x_1, y_1))) \geq 1$,

(c) $\alpha((F(x_1, y_1), F(y_1, x_1)), (Sx_1, Sy_1)) \geq 1$ and (d) $\alpha((F(y_1, x_1), F(x_1, y_1)), (Sy_1, Sx_1) \geq 1$.

From (2.1.1), there exist sequences $\{x_n\}$, $\{y_n\}$, $\{z_n\}$ and $\{w_n\}$ in X such that

$z_{2n+1} = F(x_{2n+1}, y_{2n+1}) = Tx_{2n+2}$, $w_{2n+1} = F(y_{2n+1}, x_{2n+1}) = Ty_{2n+2}$,

$z_{2n+2} = G(x_{2n+2}, y_{2n+2}) = Sx_{2n+3}$, and $w_{2n+2} = G(y_{2n+2}, x_{2n+2}) = Sy_{2n+3}$. $n = 0, 1, 2, \ldots$

From (2.1.4) (a), we have

$\alpha((Sx_1, Sy_1), (F(x_1, y_1), F(y_1, x_1))) \geq 1 \Rightarrow \alpha((Sx_1, Sy_1), (Tx_2, Ty_2)) \geq 1$, from the definition of $\{z_n\}$ and $\{w_n\}$

$\Rightarrow \alpha((F(x_1, y_1), F(y_1, x_1)), G(x_2, y_2), G(y_2, x_2))) \geq 1$, from (2.1.4)

$\Rightarrow \alpha((z_1, w_1), (z_2, w_2) \geq 1$, from the definition of $\{z_n\}$ and $\{w_n\}$

$\Rightarrow \alpha((Tx_2, Ty_2), (Sx_3, Sy_3)) \geq 1$, from the definition of $\{z_n\}$ and $\{w_n\}$

$\Rightarrow \alpha(G(x_2, y_2), G(y_2, x_2), (G(x_3, y_3), G(y_3, x_3))) \geq 1$, from (2.1.4)

$\Rightarrow \alpha((z_2, w_2), (z_3, w_3) \geq 1$.

Continuing in this way, we have $\alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1$ for all n. (1)

Similarly from (2.1.4) (c), (2.1.4) (b), (2.1.4) (d) we obtain

$\alpha((z_{n+1}, w_{n+1}), (z_n, w_n)) \geq 1$ (2)

$\alpha((w_n, z_n), (w_{n+1}, z_{n+1})) \geq 1$ (3)

$\alpha((w_{n+1}, z_{n+1}), (w_n, z_n)) \geq 1$ (4)

$R_n = \max\{ p(z_n. z_{n+1}) + \varphi(z_n) + \varphi(z_{n+1}), p(w_n. w_{n+1}) + \varphi(z_n) + \varphi(z_{n+1}) \}$.

Case(i): Suppose $R_{2m} = 0$ for some m.

Then $z_{2m} = z_{2m+1}$, $\varphi(z_{2m}) = 0$, $\varphi(z_{2m+1}) = 0$, $w_{2m} = w_{2m+1}$, $\varphi(w_{2m}) = 0$, $\varphi(w_{2m+1}) = 0$.

Now $\alpha((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2})) = \alpha((z_n, w_n), (z_{n+1}, w_{n+1})) \geq 1$ from (1).

Consider

$$\psi\left[p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+1})\right]$$
$$= \psi\left[p(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2})) + \varphi(F(x_{2m+1}, y_{2m+1}) + \varphi(G(x_{2m+2}, y_{2m+2}))\right]$$
$$\leq \alpha\left((Sx_{2m+1}, Sy_{2m+1}), (Tx_{2m+2}, Ty_{2m+2})\right)\psi\left[\begin{array}{l} p(F(x_{2m+1}, y_{2m+1}), G(x_{2m+2}, y_{2m+2})) \\ +\varphi(F(x_{2m+1}, y_{2m+1}) + \varphi(G(x_{2m+2}, y_{2m+2})) \end{array}\right]$$
$$\leq \psi\left(M^{x_{2m+2}, y_{2m+2}}_{x_{2m+1}, y_{2m+1}}\right) - \phi\left(M^{x_{2m+2}, y_{2m+2}}_{x_{2m+1}, y_{2m+1}}\right) \quad (5)$$

Where

$$M^{x_{2m+2}, y_{2m+2}}_{x_{2m+1}, y_{2m+1}} = \max\left\{\begin{array}{l} p(z_{2m}, z_{2m+1}) + \varphi(z_{2m}) + \varphi(z_{2m+1}),\ p(w_{2m}, w_{2m+1}) + \varphi(w_{2m}) + \varphi(w_{2m+1}), \\ p(z_{2m}, z_{2m}) + \varphi(z_{2m}) + \varphi(z_{2m}),\ p(w_{2m}, w_{2m}) + \varphi(w_{2m}) + \varphi(w_{2m}), \\ p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+2}),\ p(w_{2m+1}, w_{2m+2}) + \varphi(w_{2m+1}) + \varphi(w_{2m+2}), \\ \frac{1}{2}\left[p(z_{2m}, z_{2m+2}) + \varphi(z_{2m}) + \varphi(z_{2m+2}),\ p(z_{2m+1}, z_{2m+1}) + \varphi(z_{2m+1}) + \varphi(z_{2m+1})\right], \\ \frac{1}{2}\left[p(w_{2m}, w_{2m+2}) + \varphi(w_{2m}) + \varphi(w_{2m+2}),\ p(w_{2m+1}, w_{2m+1}) + \varphi(w_{2m+1}) + \varphi(w_{2m+1})\right] \end{array}\right\}$$

But

$$\frac{1}{2}\left[p(z_{2m}, z_{2m+2}) + \varphi(z_{2m}) + \varphi(z_{2m+2}) + p(z_{2m+1}, z_{2m+1}) + \varphi(z_{2m+1}) + \varphi(z_{2m+1})\right]$$
$$\leq \frac{1}{2}\left[p(z_{2m}, z_{2m+1}) + \varphi(z_{2m}) + \varphi(z_{2m+1}) + p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+2})\right]$$
$$\leq \max\left\{p(z_{2m}, z_{2m+1}) + \varphi(z_{2m}) + \varphi(z_{2m+1}),\ p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+2})\right\}.$$

Hence $M^{x_{2m+2}, y_{2m+2}}_{x_{2m+1}, y_{2m+1}} = \max\{R_{2m}, R_{2m+1}\} = R_{2m+1}$, from Case (i).

Now (5) becomes $\psi[p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+2})] \leq \psi(R_{2m+1}) - \phi(R_{2m+1})$.

Similarly using (3), we can show that $\psi[p(w_{2m+1}, w_{2m+2}) + \varphi(w_{2m+1}) + \varphi(w_{2m+2})] \leq \psi(R_{2m+1}) - \phi(R_{2m+1})$.

Thus $\psi(R_{2m+1}) \leq \psi(R_{2m+1}) - \phi(R_{2m+1})$. Which in turn yields that $\phi(R_{2m+1}) = 0$, so that

$p(z_{2m+1}, z_{2m+2}) + \varphi(z_{2m+1}) + \varphi(z_{2m+2}) = 0$ and $p(w_{2m+1}, w_{2m+2}) + \varphi(wz_{2m+1}) + \varphi(w_{2m+2}) = 0$.

Thus $z_{2m+1} = z_{2m+2}, \varphi(z_{2m+1}) = 0$, $\varphi(z_{2m+2}) = 0$ and $w_{2m+1} = w_{2m+2}$, $\varphi(w_{2m+1}) = 0$, $\varphi(w_{2m+2}) = 0$.

Continuing in this way, we get $z_{2m} = z_{2m+1} = z_{2m+2} = \ldots$ and $w_{2m} = w_{2m+1} = w_{2m+2} = \ldots$.

Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in (X, p).

Case (ii): Assume that $R_n \neq 0$ for all n.

As in Case (i), we have $\psi(R_{2n+1}) \leq \psi(\max\{R_{2n}, R_{2n+1}\}) - \phi(\max\{R_{2n}, R_{2n+1}\})$.

If $\max\{R_{2n}, R_{2n+1}\} = R_{2n+1}$, then $\psi(R_{2n+1}) \leq \psi(R_{2n+1}) - \phi(R_{2n+1})$

Which in turn yields that $R_{2n+1} = 0$.

It is contradiction to Case (ii). Thus $\psi(R_{2n+1}) \leq \psi(R_{2n}) - \phi(R_{2n})$ (6)

Similarly, we can show that $R_{2n} \leq R_{2n-1}$.

Continuing in this way, we can conclude that R_n is a non-increasing sequence of non-negative real numbers and hence must converge to a real number, say $r \geq 0$.

Suppose $r > 0$. Letting $n \to \infty$ in (6), using continuity of ψ and lower semi continuity of ϕ, we get

$\psi(r) \leq \psi(r) - \phi(r)$, Which in turn yields that $\phi(r) = 0$ so that $r = 0$.

Thus $\lim_{n\to\infty}\left[p(z_n, z_{n+1}) + \varphi(z_n) + \varphi(z_{n+1})\right] = 0$ and $\lim_{n\to\infty}\left[p(w_n, w_{n+1}) + \varphi(w_n) + \varphi(w_{n+1})\right] = 0$.

Hence $\lim_{n\to\infty} p(z_n, z_{n+1}) = 0 = \lim_{n\to\infty} p(w_n, w_{n+1})$. (7)

Hence from (p_2), we have $\lim_{n\to\infty} p(z_n, z_n) = 0 = \lim_{n\to\infty} p(w_n, w_n)$ (8)

And $\varphi(z_n) = 0 = \varphi(w_n)$. (9)

From (7), (8), definition of d_p, we have $\lim_{n\to\infty} d_p(z_n, z_{n+1}) = 0 = \lim_{n\to\infty} d_p(w_n, w_{n+1})$ (10)

Now we will prove that $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences.

On the contrary, suppose that $\{z_{2n}\}$ or $\{w_{2n}\}$ is not Cauchy, then there exist an $\varepsilon > 0$ and monotone increasing sequence of natural numbers $\{2m(k)\}$ and $\{2n(k)\}$ such that $n(k) > m(k)$

$\max\{d_p(z_{2m(k)}, z_{2n(k)}), d_p(w_{2m(k)}, w_{2n(k)})\} \geq \varepsilon$ (11)

and $\max\{d_p(z_{2m(k)}, z_{2n(k)-2}), d_p(w_{2m(k)}, w_{2n(k)-2})\} < \varepsilon$ (12)

From(11),

$$\varepsilon \leq \max\left\{d_p\left(z_{2m(k)}, z_{2n(k)}\right), d_p\left(w_{2m(k)}, w_{2n(k)}\right)\right\}$$

$$\leq \max\left\{\begin{array}{l} d_p\left(z_{2m(k)}, z_{2n(k)-2}\right) + d_p\left(z_{2n(k)-2}, z_{2n(k)-1}\right) + d_p\left(z_{2n(k)-1}, z_{2n(k)}\right), \\ d_p\left(w_{2m(k)}, w_{2n(k)-2}\right) + d_p\left(w_{2n(k)-2}, w_{2n(k)-1}\right) + d_p\left(w_{2n(k)-1}, w_{2n(k)}\right) \end{array}\right\}$$

$$\leq \max\left\{d_p\left(z_{2m(k)}, z_{2n(k)-2}\right), d_p\left(w_{2m(k)}, w_{2n(k)-2}\right)\right\} + \max\left\{\begin{array}{l} d_p\left(z_{2n(k)-2}, z_{2n(k)-1}\right) + d_p\left(z_{2n(k)-1}, z_{2n(k)}\right), \\ d_p\left(w_{2n(k)-2}, w_{2n(k)-1}\right) + d_p\left(w_{2n(k)-1}, w_{2n(k)}\right) \end{array}\right\}, \text{ from Note.}$$

Letting $k\to\infty$ and using (7) and (12), we get $\lim_{n\to\infty} \max\left\{d_p\left(z_{2m(k)}, z_{2n(k)}\right), d_p\left(w_{2m(k)}, w_{2n(k)}\right)\right\} = \varepsilon$.

Using the definition of d_p and (8), we get $\lim_{n\to\infty} \max\left\{p\left(z_{2m(k)}, z_{2n(k)}\right), p\left(w_{2m(k)}, w_{2n(k)}\right)\right\} = \frac{\varepsilon}{2}$ (13)

Letting $k \to \infty$ and using (13) and (10), in

$$\left|\max\left\{\begin{array}{l} d_p\left(z_{2m(k)+1}, z_{2n(k)}\right), \\ d_p\left(w_{2m(k)+1}, w_{2n(k)}\right) \end{array}\right\} - \max\left\{\begin{array}{l} d_p\left(z_{2m(k)}, z_{2n(k)}\right), \\ d_p\left(w_{2m(k)}, w_{2n(k)}\right) \end{array}\right\}\right| \leq \max\left\{\begin{array}{l} d_p\left(z_{2m(k)+1}, z_{2m(k)}\right), \\ d_p\left(w_{2m(k)+1}, w_{2m(k)}\right) \end{array}\right\}$$

we get $\lim_{n\to\infty} \max\left\{d_p\left(z_{2m(k)+1}, z_{2n(k)}\right), d_p\left(w_{2m(k)+1}, w_{2n(k)}\right)\right\} = \varepsilon$.

So that $\lim_{n\to\infty} \max\left\{p\left(z_{2m(k)+1}, z_{2n(k)}\right), p\left(w_{2m(k)+1}, w_{2n(k)}\right)\right\} = \frac{\varepsilon}{2}$ (14)

Letting $k \to \infty$ and using (13) and (10), in

$$\left|\max\left\{\begin{array}{l} d_p\left(z_{2m(k)+1}, z_{2n(k)-1}\right), \\ d_p\left(w_{2m(k)+1}, w_{2n(k)-1}\right)\end{array}\right\} - \max\left\{\begin{array}{l} d_p\left(z_{2m(k)}, z_{2n(k)}\right), \\ d_p\left(w_{2m(k)}, w_{2n(k)}\right)\end{array}\right\}\right| \le \max\left\{\begin{array}{l} d_p\left(z_{2m(k)+1}, z_{2m(k)}\right) + d_p\left(z_{2n(k)}, z_{2n(k)-1}\right), \\ d_p\left(w_{2m(k)+1}, w_{2m(k)}\right) + d_p\left(z_{2n(k)}, z_{2n(k)-1}\right)\end{array}\right\}$$

we get $\lim_{n\to\infty} \max\left\{d_p\left(z_{2m(k)+1}, z_{2n(k)-1}\right), d_p\left(w_{2m(k)+1}, w_{2n(k)-1}\right)\right\} = \varepsilon$.

So that $\lim_{n\to\infty} \max\left\{p\left(z_{2m(k)+1}, z_{2n(k)-1}\right), p\left(w_{2m(k)+1}, w_{2n(k)-1}\right)\right\} = \dfrac{\varepsilon}{2}$ (15)

Letting $k \to \infty$ and using (13) and (10), in

$$\left|\max\left\{\begin{array}{l} d_p\left(z_{2m(k)}, z_{2n(k)-1}\right), \\ d_p\left(w_{2m(k)}, w_{2n(k)-1}\right)\end{array}\right\} - \max\left\{\begin{array}{l} d_p\left(z_{2m(k)}, z_{2n(k)}\right), \\ d_p\left(w_{2m(k)}, w_{2n(k)}\right)\end{array}\right\}\right| \le \max\left\{\begin{array}{l} d_p\left(z_{2n(k)}, z_{2n(k)-1}\right), \\ d_p\left(w_{2n(k)}, w_{2n(k)-1}\right)\end{array}\right\}$$

we get $\lim_{n\to\infty} \max\left\{d_p\left(z_{2m(k)}, z_{2n(k)-1}\right), d_p\left(w_{2m(k)}, w_{2n(k)-1}\right)\right\} = \varepsilon$.

So that $\lim_{n\to\infty} \max\left\{p\left(z_{2m(k)}, z_{2n(k)-1}\right), p\left(w_{2m(k)}, w_{2n(k)-1}\right)\right\} = \dfrac{\varepsilon}{2}$ (16)

Now by using (1) and triangular property of α, we have

Now $\alpha((Sx_{2m(k)+1}, Sy_{2m(k)+1}), (Tx_{2n(k)}, Ty_{2n(k)})) = \alpha((z_{2m(k)}, w_{2m(k)}), (z_{2n(k)-1}, w_{2n(k)-1})) \ge 1$.

Consider

$$\begin{aligned}
&\psi\left[p(z_{2m(k)+1}, z_{2n(k)}) + \varphi(z_{2m(k)+1}) + \varphi(z_{2n(k)})\right] \\
&\quad = \psi\left[p(F(x_{2m(k)+1}, y_{2m(k)+1}), G(x_{2n(k)}, y_{2n(k)})) + \varphi(F(x_{2m(k)+1}, y_{2m(k)+1}) + \varphi(G(x_{2n(k)}, y_{2n(k)}))\right] \\
&\quad \le \alpha\left(\left(Sx_{2m(k)+1}, Sy_{2m(k)+1}\right), \left(Tx_{2n(k)}, Ty_{2n(k)}\right)\right)\psi\left[\begin{array}{l} p(F(x_{2m(k)+1}, y_{2m(k)+1}), G(x_{2n(k)}, y_{2n(k)})) \\ +\varphi(F(x_{2m(k)+1}, y_{2m(k)+1}) + \varphi(G(x_{2n(k)}, y_{2n(k)})) \end{array}\right] \\
&\quad \le \psi\left(M^{x_{2n(k)}, y_{2n(k)}}_{x_{2m(k)+1}, y_{2m(k)+1}}\right) - \phi\left(M^{x_{2n(k)}, y_{2n(k)}}_{x_{2m(k)+1}, y_{2m(k)+1}}\right)
\end{aligned}$$

where

$$M^{x_{2n(k)}, y_{2n(k)}}_{x_{2m(k)+1}, y_{2m(k)+1}} = \max\left\{\begin{array}{l}
p(z_{2m(k)}, z_{2n(k)-1}) + \varphi(z_{2m(k)}) + \varphi(z_{2n(k)-1}),\ p(w_{2m(k)}, w_{2n(k)-1}) + \varphi(w_{2m(k)}) + \varphi(w_{2n(k)-1}), \\
p(z_{2m(k)}, z_{2m(k)+1}) + \varphi(z_{2m(k)}) + \varphi(z_{2m(k)+1}),\ p(w_{2m(k)}, w_{2m(k)+1}) + \varphi(w_{2m(k)}) + \varphi(w_{2m(k)+1}), \\
p(z_{2n(k)-1}, z_{2n(k)}) + \varphi(z_{2n(k)-1}) + \varphi(z_{2n(k)}),\ p(w_{2n(k)-1}, w_{2n(k)}) + \varphi(w_{2n(k)-1}) + \varphi(w_{2n(k)}), \\
\frac{1}{2}\left[p(z_{2m(k)}, z_{2n(k)}) + \varphi(z_{2m(k)}) + \varphi(z_{2n(k)}),\ p(z_{2n(k)-1}, z_{2m(k)+1}) + \varphi(z_{2n(k)-1}) + \varphi(z_{2m(k)+1}),\ \right], \\
\frac{1}{2}\left[p(w_{2m(k)}, w_{2n(k)}) + \varphi(w_{2m(k)}) + \varphi(w_{2n(k)}),\ p(w_{2n(k)-1}, w_{2m(k)+1}) + \varphi(w_{2n(k)-1}) + \varphi(w_{2m(k)+1})\ \right]
\end{array}\right\}$$

$$\to \max\left\{\frac{\varepsilon}{2}, 0, 0, \frac{1}{2}\left[\frac{\varepsilon}{2} + \frac{\varepsilon}{2}\right]\right\} \text{as } k \to \infty = \frac{\varepsilon}{2}, \text{ from (7), (13), (15) and (16).}$$

Thus $\lim_{n\to\infty} \psi\left[p\left(z_{2m(k)+1}, z_{2n(k)}\right) + \varphi\left(z_{2m(k)+1}\right) + \varphi\left(z_{2n(k)}\right)\right] \le \psi\left(\dfrac{\varepsilon}{2}\right) - \phi\left(\dfrac{\varepsilon}{2}\right)$

and $\lim_{n\to\infty} \psi\left[p\left(w_{2m(k)+1}, w_{2n(k)}\right) + \varphi\left(w_{2m(k)+1}\right) + \varphi\left(w_{2n(k)}\right)\right] \le \psi\left(\dfrac{\varepsilon}{2}\right) - \phi\left(\dfrac{\varepsilon}{2}\right)$.

From (14), we have $\psi\left(\dfrac{\varepsilon}{2}\right) \le \psi\left(\dfrac{\varepsilon}{2}\right) - \phi\left(\dfrac{\varepsilon}{2}\right)$

which in turn yields that $\phi\left(\dfrac{\varepsilon}{2}\right) = 0$ so that $\varepsilon = 0$. It is contradiction.

Hence $\{z_{2n}\}$ and $\{w_{2n}\}$ are Cauchy sequences. Letting $n, m \to \infty$ and using (7) and (13), in

$$\left| \max\left\{ \begin{array}{l} d_p(z_{2n+1}, z_{2m+1}), \\ d_p(w_{2n+1}, w_{2m+1}) \end{array} \right\} - \max\left\{ \begin{array}{l} d_p(z_{2n}, z_{2m}), \\ d_p(w_{2n}, w_{2m}) \end{array} \right\} \right| \leq \max\left\{ \begin{array}{l} d_p(z_{2n+1}, z_{2n}) + d_p(z_{2m}, z_{2m+1}), \\ d_p(w_{2n+1}, w_{2n}) + d_p(z_{2m}, z_{2m+1}) \end{array} \right\}$$

we get $\lim_{n, m \to \infty} d_p(z_{2n+1}, z_{2m+1}) = 0 = \lim_{n, m \to \infty} d_p(w_{2n+1}, w_{2m+1})$.

Thus $\{z_{2m+1}\}$ and $\{w_{2m+1}\}$ are Cauchy. Hence $\{z_n\}$ and $\{w_n\}$ are Cauchy. $\lim_{n, m \to \infty} d_p(z_n, z_m) = 0 = \lim_{n, m \to \infty} d_p(w_n, w_m)$.

From the definition of d_p and (8), it follows that $\lim_{n, m \to \infty} p(z_n, z_m) = 0 = \lim_{n, m \to \infty} p(w_n, w_m)$ (17)

Thus $\{z_n\}$ and $\{w_n\}$ are Cauchy.

Suppose S(X) is complete. Since $\{z_{2n+2}\} \subseteq S(X)$ and $\{w_{2n+2}\} \subseteq S(X)$ are Cauchy sequences in the complete metric space $(S(X), d_p)$, it follows that the sequences $\{z_{2n+2}\}$ and $\{w_{2n+2}\}$ are convergent in $(S(X), d_p)$. Thus $\lim_{n\to\infty} d_p(z_{2n+2}, u) = 0$ and $\lim_{n\to\infty} d_p(w_{2n+2}, v) = 0$ for some u, v in S(X). Since $u, v \in S(X)$ there exist $l, m \in X$ such that

$u = Sl$ and $v = Sm$. Since $\{z_n\}$ and $\{w_n\}$ are Cauchy sequences in X and $\{z_{2n+2}\} \to u$ and $\{w_{2n+2}\} \to v$, it follows that

$\{z_{2n+1}\} \to u$ and $\{w_{2n+1}\} \to v$. Since φ is lower semi continuous, we have

$$\varphi(u) \leq \lim_{n\to\infty} \inf \varphi(z_n) \leq \lim_{n\to\infty} \varphi(z_n) = 0 \text{ and } \varphi(v) \leq \lim_{n\to\infty} \inf \varphi(w_n) \leq \lim_{n\to\infty} \varphi(w_n) = 0 .$$

Hence $\varphi(u) = 0$ and $\varphi(v) = 0$. (18)

From Lemma 1.4 (ii), we have

$$p(u, u) = \lim_{n\to\infty} p(z_{2n+1}, u) = \lim_{n\to\infty} p(z_{2n+2}, u) = \lim_{n,m \to \infty} p(z_n, z_m) \tag{19}$$

$$p(v, v) = \lim_{n\to\infty} p(w_{2n+1}, v) = \lim_{n\to\infty} p(w_{2n+2}, v) = \lim_{n,m \to \infty} p(w_n, w_m) \tag{20}$$

From (19), (20) and (17), we have $p(u, u) = 0 = p(v, v)$ (21)

From (2.1.7)(i), $\alpha((Sl, Sm), (Tx_{2n+2}, Ty_{2n+2})) = \alpha((u, v),(z_{2n+1}, w_{2n+1})) \geq 1$.

Consider

$$\begin{aligned} &\psi\left[p(F(l, m), G(x_{2n+2}, y_{2n+2})) + \varphi(F(l, m)) + \varphi(G(x_{2n+2}, y_{2n+2}))\right] \\ &\quad \leq \alpha\left((Sl, Sm), (Tx_{2n+2}, Ty_{2n+2})\right)\psi\left[\begin{array}{l} p(F(l, m), G(x_{2n+2}, y_{2n+2})) \\ + \varphi(F(l, m)) + \varphi(G(x_{2n+2}, y_{2n+2})) \end{array}\right] \\ &\quad \leq \psi\left(M_{l, m}^{x_{2n+2}, y_{2n+2}}\right) - \phi\left(M_{l, m}^{x_{2n+2}, y_{2n+2}}\right) \end{aligned} \tag{22}$$

Where

$$M_{l, m}^{x_{2n+2}, y_{2n+2}} = \max\left\{ \begin{array}{l} p(u, z_{2n+1}) + \varphi(u) + \varphi(z_{2n+1}), \ p(v, w_{2n+1}) + \varphi(v) + \varphi(w_{2n+1}), \\ p(u, F(l, m)) + \varphi(u) + \varphi(F(l, m)), \ p(v, F(m, l)) + \varphi(v) + \varphi(F(m, l)), \\ p(z_{2n+1}, z_{2n+2}) + \varphi(z_{2n+1}) + \varphi(z_{2n+2}), \ p(w_{2n+1}, w_{2n+2}) + \varphi(w_{2n+1}) + \varphi(w_{2n+2}), \\ \frac{1}{2}[p(u, z_{2n+2}) + \varphi(u) + \varphi(z_{2n+2}) + p(z_{2n+1}, F(l, m)) + \varphi(z_{2n+1}) + \varphi(F(l, m)),], \\ \frac{1}{2}[p(v, w_{2n+2}) + \varphi(v) + \varphi(w_{2n+2}) + p(w_{2n+1}, F(m, l)) + \varphi(w_{2n+1}) + \varphi(F(m, l))] \end{array} \right\}$$

$\to \max\{p(u, F(l, m)) + \varphi(F(l, m)), \ p(v, F(m, l)) + \varphi(F(m, l))\}$ from (9), (18), (19), (20) and Lemma 1.5

Now letting $n \to \infty$ in (22) and using (9) and Lemma 1.5, we get

$$\psi\left(p(F(l, m), u) + \varphi(F(l, m))\right) \le \psi\left(\max\left\{p(u, F(l, m)) + \varphi(F(l, m)), p(v, F(m, l)) + \varphi(F(m, l))\right\}\right)$$
$$- \phi\left(\max\left\{p(u, F(l, m)) + \varphi(F(l, m)), p(v, F(m, l)) + \varphi(F(m, l))\right\}\right).$$

Similarly by using (2.1.7) (i), we get

$$\psi\left(p(F(m, l), v) + \varphi(F(m, l))\right) \le \psi\left(\max\left\{p(u, F(l, m)) + \varphi(F(l, m)), p(v, F(m, l)) + \varphi(F(m, l))\right\}\right)$$
$$-\phi\left(\max\left\{p(u, F(l, m)) + \varphi(F(l, m)), p(v, F(m, l)) + \varphi(F(m, l))\right\}\right).$$

Thus $\psi\left(\max\left\{\begin{matrix} p(u, F(l, m)) + \varphi(F(l, m)), \\ p(v, F(m, l)) + \varphi(F(m, l)) \end{matrix}\right\}\right) \le \psi\left(\max\left\{\begin{matrix} p(u, F(l, m)) + \varphi(F(l, m)), \\ p(v, F(m, l)) + \varphi(F(m, l)) \end{matrix}\right\}\right) - \phi\left(\max\left\{\begin{matrix} p(u, F(l, m)) + \varphi(F(l, m)), \\ p(v, F(m, l)) + \varphi(F(m, l)) \end{matrix}\right\}\right).$

which turn yields that $\phi\left(\max\left\{\begin{matrix} p(u, F(l, m)) + \varphi(F(l, m)), \\ p(v, F(m, l)) + \varphi(F(m, l)) \end{matrix}\right\}\right) = 0.$

Thus $p(F(l, m), u) = 0$, $\varphi(F(l, m)) = 0$ and $p(F(m, l), v) = 0$, $\varphi(F(m, l)) = 0$.

Hence $F(l, m) = u$ and $F(m, l) = v$. Hence $Sl = u = F(l, m)$. Similarly $Sm = v = F(m, l)$.

Since the pair (F, S) is w-compatible, we have

$Su = S(Sl) = S(F(l, m)) = F(Sl, Sm) = F(u, v)$ and $Sv = S(Sm) = S(F(m, l)) = F(Sm, Sl) = F(v, u)$.

Now from (2.1.7) (ii), we have

$\alpha((Su, Sv), (Tx_{2n+2}, Ty_{2n+2})) = \alpha((Su, Sv), (z_{2n+1}, w_{2n+1})) \ge 1$. Consider

$$\begin{aligned} &\psi\left[p(F(u, v), G(x_{2n+2}, y_{2n+2})) + \varphi(F(u, v)) + \varphi(G(x_{2n+2}, y_{2n+2}))\right] \\ &\le \alpha\left((Su, Sv), (Tx_{2n+2}, Ty_{2n+2})\right)\psi\left[\begin{matrix} p(F(u, v), G(x_{2n+2}, y_{2n+2})) \\ + \varphi(F(u, v)) + \varphi(G(x_{2n+2}, y_{2n+2})) \end{matrix}\right] \\ &\le \psi\left(M_{u,v}^{x_{2n+2}, y_{2n+2}}\right) - \phi\left(M_{u,v}^{x_{2n+2}, y_{2n+2}}\right) \qquad (23) \end{aligned}$$

Where

$$M_{l,m}^{x_{2n+2}, y_{2n+2}} = \max\left\{\begin{matrix} p(Su, z_{2n+1}) + \varphi(Su) + \varphi(z_{2n+1}), \ p(Sv, w_{2n+1}) + \varphi(Sv) + \varphi(w_{2n+1}), \\ p(Su, Su) + \varphi(Su) + \varphi(Su), \ p(Sv, Sv) + \varphi(Sv) + \varphi(Sv), \\ p(z_{2n+1}, z_{2n+2}) + \varphi(z_{2n+1}) + \varphi(z_{2n+2}), \ p(w_{2n+1}, w_{2n+2}) + \varphi(w_{2n+1}) + \varphi(w_{2n+2}), \\ \frac{1}{2}[p(Su, z_{2n+2}) + \varphi(Su) + \varphi(z_{2n+2}) + p(z_{2n+1}, Su) + \varphi(z_{2n+1}) + \varphi(Su)),], \\ \frac{1}{2}[p(Sv, w_{2n+2}) + \varphi(Sv) + \varphi(w_{2n+2}) + p(w_{2n+1}, Sv) + \varphi(w_{2n+1}) + \varphi(Sv)] \end{matrix}\right\}$$

$$\to \max\left\{\begin{matrix} p(Su, u) + \varphi(Su), \ p(Sv, v) + \varphi(Sv), \ 2\varphi(Su), \ 2\varphi(Sv), 0, 0, \\ \frac{1}{2}[p(Su, u) + \varphi(Su) + p(u, Su) + \varphi(Su)),], \frac{1}{2}[p(Sv, v) + \varphi(Sv) + p(v, Sv) + \varphi(Sv)] \end{matrix}\right\} \text{as } n \to \infty$$

$$\to \max\left\{p(Su, u) + \varphi(Su), \ p(Sv, v) + \varphi(Sv)\right\}.$$

Now letting $n \to \infty$ in (22) and using (9) and Lemma 1.5, we get

$$\psi\left(p(Su, u) + \varphi(Su)\right) \le \psi\left(\max\left\{p(Su, u) + \varphi(Su), p(Sv, v) + \varphi(Sv)\right\}\right)$$
$$- \phi\left(\max\left\{p(Su, u) + \varphi(Su), p(Sv, v) + \varphi(Sv)\right\}\right).$$

Similarly by using (2.1.7) (ii), we get

$$\psi\left(p(Sv, v) + \varphi(Sv)\right) \leq \psi\left(\max\{p(Su, u) + \varphi(Su), p(Sv, v) + \varphi(Sv)\}\right)$$
$$- \phi\left(\max\{p(Su, u) + \varphi(Su), p(Sv, v) + \varphi(Sv)\}\right).$$

Thus $\psi\begin{pmatrix} p(Su, u) + \varphi(Su), \\ p(Sv, v) + \varphi(Sv) \end{pmatrix} \leq \psi\left(\max\begin{Bmatrix} p(Su, u) + \varphi(Su), \\ p(Sv, v) + \varphi(Sv) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(Su, u) + \varphi(Su), \\ p(Sv, v) + \varphi(Sv) \end{Bmatrix}\right).$

which turn yields that $\phi\left(\max\{p(Su, u) + \varphi(Su), p(Sv, v) + \varphi(Sv)\}\right) = 0.$

Thus p(Su, u) = 0, φ(Su) = 0 and p(Sv, v) = 0, φ(Sv) = 0. Hence Su = u and Sv = v. Then F(u, v) = Su = u (24)

and F(v, u) = Sv = v. (25)

Since $F(X \times X) \subseteq T(X)$, there exist α, β in X such that Tα = F(u, v) = Su = u and Tβ = F(v, u) = Sv = v.

Now α((Su, Sv), (Tα, Tβ)) = α((u, v), (u, v)) ≥ 1 from (2.1.7) (iii)

Consider

$$\psi\left[p(T\alpha, G(\alpha, \beta)) + \varphi(T\alpha) + \varphi(G(\alpha, \beta))\right]$$
$$\leq \alpha\left((Su, Sv), (T\alpha, T\beta)\right)\psi\left[p(F(u, v), G(\alpha, \beta)) + \varphi(F(u, v)) + \varphi(G(\alpha, \beta))\right]$$
$$\leq \psi\left(M_{u,v}^{\alpha,\beta}\right) - \phi\left(M_{u,v}^{\alpha,\beta}\right) \quad (26)$$

Where

$$M_{u,v}^{\alpha,\beta} = \max\begin{Bmatrix} 0, 0, 0, 0, p(T\alpha, G(\alpha, \beta)) + \varphi(T\alpha) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(T\beta) + \varphi(G(\beta, \alpha)), \\ \frac{1}{2}[p(T\alpha, G(\alpha, \beta)) + \varphi(T\alpha) + \varphi(G(\alpha, \beta)) + 0], \\ \frac{1}{2}[p(T\beta, G(\beta, \alpha)) + \varphi(T\beta) + \varphi(G(\beta, \alpha)) + 0] \end{Bmatrix}$$
$$= \max\begin{Bmatrix} p(T\alpha, G(\alpha, \beta)) + \varphi(T\alpha) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(T\beta) + \varphi(G(\beta, \alpha)) \end{Bmatrix}$$

Now (26) becomes,

$$\psi\left(p(T\alpha, G(\alpha, \beta)) + \varphi(G(\alpha, \beta))\right) \leq \psi\left(\max\begin{Bmatrix} p(T\alpha, G(\alpha, \beta)) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(G(\beta, \alpha)) \end{Bmatrix}\right)$$
$$-\phi\left(\max\begin{Bmatrix} p(T\alpha, G(\alpha, \beta)) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(G(\beta, \alpha)) \end{Bmatrix}\right).$$

Similarly by using (2.1.7) (iii), we have

$$\psi\left(p(T\beta, G(\beta, \alpha)) + \varphi(G(\beta, \alpha))\right) \leq \psi\left(\max\begin{Bmatrix} p(T\alpha, G(\alpha, \beta)) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(G(\beta, \alpha)) \end{Bmatrix}\right)$$
$$-\phi\left(\max\begin{Bmatrix} p(T\alpha, G(\alpha, \beta)) + \varphi(G(\alpha, \beta)), \\ p(T\beta, G(\beta, \alpha)) + \varphi(G(\beta, \alpha)) \end{Bmatrix}\right).$$

Thus

$$\psi\begin{pmatrix}p(T\alpha, G(\alpha,\beta)) + \varphi(G(\alpha,\beta)),\\ p(T\beta, G(\beta,\ \alpha)) + \varphi(G(\beta,\ \alpha))\end{pmatrix} \le \psi\left(\max\begin{Bmatrix}p(T\alpha, G(\alpha,\beta)) + \varphi(G(\alpha,\beta)),\\ p(T\beta, G(\beta,\ \alpha)) + \varphi(G(\beta,\ \alpha))\end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix}p(T\alpha, G(\alpha,\beta)) + \varphi(G(\alpha,\beta)),\\ p(T\beta, G(\beta,\ \alpha)) + \varphi(G(\beta,\ \alpha))\end{Bmatrix}\right)$$

which in turn yields that $\phi\left(\max\begin{Bmatrix}p(T\alpha, G(\alpha,\beta)) + \varphi(G(\alpha,\beta)),\\ p(T\beta, G(\beta,\ \alpha)) + \varphi(G(\beta,\ \alpha))\end{Bmatrix}\right) = 0.$

Thus $p(T\alpha, G(\alpha, \beta)) = 0$, $\varphi(G(\alpha, \beta)) = 0$ and $p(T\beta, G(\beta, \alpha)) = 0$, $\varphi(G(\beta, \alpha)) = 0$.

Hence $T\alpha = G(\alpha, \beta)$ and $T\beta = G(\beta, \alpha)$.

Since the pair (G, T) is w-compatible, we have $Tu = T(T\alpha) = T(G(\alpha, \beta)) = G(T\alpha, T\beta) = G(u, v)$

and $Tv = T(T\beta) = T(G(\beta, \alpha)) = G(T\beta, T\alpha) = G(v, u)$.

Now $\alpha((Su, Sv), (Tu, Tv)) = \alpha((u, v), (Tu, Tv)) \ge 1$ from (2.1.7) (iv).

Consider

$$\begin{aligned}\psi\left[p(u, G(u, v)) + \varphi(u)+\varphi(G(u, v))\right] &\le \alpha\big((Su, Sv), (Tu, Tv)\big)\psi\left[p(F(u, v), G(u, v)) + \varphi(F(u, v))+\varphi(G(u, v))\right]\\ &\le \psi\left(M_{u,v}^{u,v}\right)-\phi\left(M_{u,v}^{u,v}\right) \qquad (27)\end{aligned}$$

Where

$$M_{u,v}^{u,v} = \max\begin{Bmatrix} p(u, G(u, v)) + \varphi(u) + \varphi(G(u, v)),\ p(v, G(v, u)) + \varphi(v) + \varphi(G(v, u)),\\ p(u, u)+\varphi(u)+\varphi(u),\ p(v, v)+\varphi(v)+\varphi(v),\\ p(G(u, v), G(u, v)) + \varphi(G(u, v)) + \varphi(G(u, v)),\ p(G(v, u), G(v, u)) + \varphi(G(v, u)) + \varphi(G(v, u)),\\ \frac{1}{2}\left[p(u, G(u, v)) + \varphi(u) + \varphi(G(u, v)) + p(G(u, v), u) + \varphi(G(u, v)) + \varphi(u)\right],\\ \frac{1}{2}\left[p(v, G(v, u)) + \varphi(v) + \varphi(G(v, u)) + p(G(v, u), v) + \varphi(G(v, u)) + \varphi(v)\right]\end{Bmatrix}$$

$$= \max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)),\ p(v, G(v, u)) + \varphi(G(v, u)),\ 0,\ 0,\ 2\varphi(G(u, v)),\ 2\varphi(G(v, u)),\\ \frac{1}{2}\left[p(u, G(u, v)) + \varphi(G(u, v)) + p(G(u, v), u) + \varphi(G(u, v))\right],\\ \frac{1}{2}\left[p(v, G(v, u)) + \varphi(G(v, u)) + p(G(v, u), v) + \varphi(G(v, u))\right]\end{Bmatrix}$$

$$= \max\left\{p(u, G(u, v)) + \varphi(G(u, v)),\ p(v, G(v, u)) + \varphi(G(v, u))\right\}.$$

Now (27) becomes,

$$\psi\left(p(u, G(u, v)) + \varphi(G(u, v))\right) \le \psi\left(\max\begin{Bmatrix}p(u, G(u, v)) + \varphi(G(u, v)),\\ p(v, G(v, u)) + \varphi(G(v, u))\end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix}p(u, G(u, v)) + \varphi(G(u, v)),\\ p(v, G(v, u)) + \varphi(G(v, u))\end{Bmatrix}\right).$$

Similarly by using (2.1.7) (iv), we have

$$\psi\left(p(v, G(v, u)) + \varphi(G(v, u))\right) \le \psi\left(\max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{Bmatrix}\right).$$

Thus

$$\psi\begin{pmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{pmatrix} \le \psi\left(\max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{Bmatrix}\right).$$

which in turn yields that $\phi\left(\max\begin{Bmatrix} p(u, G(u, v)) + \varphi(G(u, v)), \\ p(v, G(v, u)) + \varphi(G(v, u)) \end{Bmatrix}\right) = 0.$

Hence $p(u, G(u, v)) = 0$, $\varphi(G(u, v)) = 0$ and $p(v, G(v, u)) = 0$, $\varphi(G(v, u)) = 0$.

Thus $u = G(u, v) = Tu$ (28)

$v = G(v, u) = Tv$ (29)

From (24), (25), (28) and (29), it follows that (u, v) is a common coupled fixed point of F, G, S and T.

Suppose (u^1, v^1) is another common coupled fixed point of F, G, S and T.

Now $\alpha((Su, Sv),(Tu^1,Tv^1)) = \alpha((u,v),(u^1,v^1)) \ge 1$ from (2.1.8).

$$\begin{aligned} &\psi\left[p(u, u^1) + \varphi(u) + \varphi(u^1)\right] \\ &\le \alpha\left((Su, Sv), (Tu^1, Tv^1)\right)\psi\left[p(F(u, v), G(u^1, v^1)) + \varphi(F(u, v)) + \varphi(G(u^1, v^1))\right] \\ &\le \psi\left(M_{u, v}^{u^1, v^1}\right) - \phi\left(M_{u, v}^{u^1, v^1}\right) \qquad (30) \end{aligned}$$

Where

$$M_{u, v}^{u^1, v^1} = \max\begin{Bmatrix} p(u, u^1) + \varphi(u) + \varphi(u^1), p(v, v^1) + \varphi(v) + \varphi(v^1), p(u, u) + \varphi(u) + \varphi(u), \\ p(v,v) + \varphi(v) + \varphi(v), p(u^1,u^1) + \varphi(u^1) + \varphi(u^1), p(v^1,v^1) + \varphi(v^1) + \varphi(v^1), \\ \frac{1}{2}\left[p(u, u) + \varphi(u) + \varphi(u) + p(u, u) + \varphi(u) + \varphi(u)\right], \\ \frac{1}{2}\left[p(v,v) + \varphi(v) + \varphi(v) + p(v,v) + \varphi(v) + \varphi(v)\right] \end{Bmatrix}$$

$$= \max\left\{p(u, u^1) + \varphi(u^1), p(v, v^1) + \varphi(v^1)\right\}.$$

Now (30) becomes, $\psi\left(p(u, u^1) + \varphi(u^1)\right) \le \psi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right).$

Similarly by using (2.1.8), we get

$$\psi\left(p(v, v^1) + \varphi(v^1)\right) \le \psi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right).$$

Thus $\psi\begin{pmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{pmatrix} \le \psi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right) - \phi\left(\max\begin{Bmatrix} p(u, u^1) + \varphi(u^1), \\ p(v, v^1) + \varphi(v^1) \end{Bmatrix}\right).$

Which in turn yields that $\phi\left(\max\left\{p(u, u^1) + \varphi(u^1), p(v, v^1) + \varphi(v^1)\right\}\right) = 0.$

$p(u, u^1) = 0$, $\varphi(u^1) = 0$ and $p(v, v^1) = 0$, $\varphi(v^1) = 0$. Hence $u = u^1$ and $v = v^1$.

Thus (u, v) is unique common coupled fixed point of F, G, S and T.

Now we give an example to illustrate our main Theorem 2.1.

3. Example

Example 3.1: Let $X = [0, 2]$ and $p(x, y) = \max\{x, y\}$ for all $x, y \in X$. Define $F, G : X \times X \to X$ by $F(x, y) = \frac{x^2 + y^2}{16}$ and $G(x, y) = \frac{x^2 + y^2}{24}$ and $S, T : X \to X$ by $Sx = \frac{x^2}{2}$ and $Tx = \frac{x^2}{3}$.

Define $\alpha : X^2 \times X^2 \to R^+$ by $\alpha\left((x, y), (u, v)\right) = \begin{cases} 1, & \text{if } x, y, u, v \in [0, \sqrt{3}] \\ 2, & \text{otherwise} \end{cases}$

Let $\psi, \phi, \varphi : R^+ \to R^+$ be defined by $\psi(t) = t$, $\phi(t) = \frac{t}{2}$ and $\varphi(t) = t$.

Clearly $F(X \times X) \subseteq T(X)$ and $G(X \times X) \subseteq S(X)$.

To verify (2.1.3) we consider the following two cases.

Case (a) : Suppose $x, y, u, v \in [0, \sqrt{3}]$.

Then $\alpha\left((Sx, Sy), (Tu, Tv)\right) = \alpha\left(\left(\frac{x^2}{2}, \frac{y^2}{2}\right), \left(\frac{u^2}{2}, \frac{v^2}{2}\right)\right) = 1.$

$$\alpha\left((Sx, Sy), (Tu, Tv)\right)\psi\left(p\left(F(x, y), G(u, v)\right) + \varphi(F(x, y)) + \varphi(G(u, v))\right)$$

$$= \max\left\{\frac{x^2 + y^2}{16}, \frac{u^2 + v^2}{24}\right\} + \frac{x^2 + y^2}{16} + \frac{u^2 + v^2}{24}$$

$$= \frac{1}{8}\left[\max\left\{\frac{x^2 + y^2}{2}, \frac{u^2 + v^2}{3}\right\} + \frac{x^2 + y^2}{2} + \frac{u^2 + v^2}{3}\right]$$

$$\leq \frac{1}{8}\left[\max\left\{\frac{x^2}{2}, \frac{u^2}{3}\right\} + \max\left\{\frac{y^2}{2}, \frac{v^2}{3}\right\} + \left(\frac{x^2}{2} + \frac{u^2}{3}\right) + \left(\frac{y^2}{2} + \frac{v^2}{3}\right)\right]$$

$$= \frac{1}{8}\left[p(Sx, Tu) + \varphi(Sx) + \varphi(Tu) + p(Sy, Tv) + \varphi(Sy) + \varphi(Tv)\right]$$

$$\leq \frac{2}{8}\left[\max\left\{p(Sx, Tu) + \varphi(Sx) + \varphi(Tu), p(Sy, Tv) + \varphi(Sy) + \varphi(Tv)\right\}\right]$$

$$\leq \frac{1}{4}M^{u,v}_{x,y} \leq \frac{1}{2}M^{u,v}_{x,y} = \psi\left(M^{u,v}_{x,y}\right) - \phi\left(M^{u,v}_{x,y}\right).$$

Case (b): $x, y, u, v \notin [0, \sqrt{3}]$. Then $\alpha\left((Sx, Sy), (Tu, Tv)\right) = \alpha\left(\left(\frac{x^2}{2}, \frac{y^2}{2}\right), \left(\frac{u^2}{2}, \frac{v^2}{2}\right)\right).$

If $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3}$ in $\left[1.5, \sqrt{3}\right]$ then $\alpha((Sx, Sy), (Tu, Tv)) = 1.$

The inequality (2.1.3) holds as in Case (a).

If at least one of $\frac{x^2}{2}, \frac{y^2}{2}, \frac{u^2}{3}, \frac{v^2}{3}$ in $\left[\sqrt{3}, 2\right]$ then $\alpha((Sx, Sy), (Tu, Tv)) = 2.$ Then

$$\alpha\left((Sx, Sy), (Tu, Tv)\right)\psi\left(p\left(F(x, y), G(u, v)\right) + \varphi(F(x, y)) + \varphi(G(u, v))\right)$$
$$= 2\max\left\{\frac{x^2 + y^2}{16}, \frac{u^2 + v^2}{24}\right\} + \frac{x^2 + y^2}{16} + \frac{u^2 + v^2}{24}$$
$$= \frac{2}{8}\left[\max\left\{\frac{x^2 + y^2}{2}, \frac{u^2 + v^2}{3}\right\} + \frac{x^2 + y^2}{2} + \frac{u^2 + v^2}{3}\right]$$
$$\leq \frac{2}{8}\left[\max\left\{\frac{x^2}{2}, \frac{u^2}{3}\right\} + \max\left\{\frac{y^2}{2}, \frac{v^2}{3}\right\} + \left(\frac{x^2}{2} + \frac{u^2}{3}\right) + \left(\frac{y^2}{2} + \frac{v^2}{3}\right)\right]$$
$$= \frac{2}{8}\left[p(Sx, Tu) + \varphi(Sx) + \varphi(Tu) + p(Sy, Tv) + \varphi(Sy) + \varphi(Tv)\right]$$
$$\leq \frac{4}{8}\left[\max\left\{p(Sx, Tu) + \varphi(Sx) + \varphi(Tu), p(Sy, Tv) + \varphi(Sy) + \varphi(Tv)\right\}\right]$$
$$\leq \frac{1}{2}M_{x, y}^{u, v} = \psi\left(M_{x, y}^{u, v}\right) - \phi\left(M_{x, y}^{u, v}\right).$$

By definition of α one can easily verify the conditions of (2.1.7) and (2.1.8). Thus the all conditions of Theorem 2.1 are satisfied and (0, 0) is the unique common coupled fixed point of F, G, S and T.

References

[1] Abbas, M., Ali khan, M., Radenovic, S., Common coupled fixed point theorems on cone metric spaces for w- compatible mappings, Applied Mathematics and Computation,Volume 217, Issue 1, Sept 2010, Pages 195-202.

[2] Abdeljawad, T., Karapinar, E. and Tas, K., Existance and uniqueness of a common fixed point on partial metric spaces, Appl. Math. Lett., 24(11) (2011), 1900-1904.

[3] Alber, Ya.I. and Guerre-delabriere, S., Principles of weakly contractive maps in Hilbert spaces, New Results in Operator Theory, Advances and Appl., 98, (1997), 7-22.

[4] Altun, I. and Erduran, A., Fixed point theorems for monotone mappings on partial metric spaces, Fixed point theory Appl., 2011(2011), Article ID 736063, 9 pages.

[5] Altun, I., Sola Simsek, F., Generalized contractions on partial metric spaces, Topology Appl., 157 (2010), No.18, 2778-2785.

[6] Aydi, H., On common fixed point theorems for ψ, φ - generalized f-weakly contractive mappings, Miskolc Math. Notes,14(2013), 19 - 30.

[7] Bhaskar, T. G. and Lakshmikantham, V., Fixed point theorems in partially ordered metric spaces and applications. Nonlinear Analysis 65 (2006), 1379-1393.

[8] Karapinar , E. and Erhan, I.M., Fixed point theorems for operators on partial metric spaces, Appl.Math.Lett., 24(2011), No.11, 1894-1899.

[9] Karapinar, E., Generalizations of Caristi Kirk's Theorem on Partial metric spaces, Fixed point theory Appl., 2011(2011:4), 7 pages.

[10] Kaushik, P., Kumar S. and Kumam, P., Coupled coincidence fixed point theorems for α, ψ - contractive type mappings in partially ordered metric spaces, Fixed Point Theory and Applications, 2013, 2013:325, 10 pages.

[11] Khan, M.S., Rao K.P.R. and Parvathi, K.V.S., A Unique Common Coupled fixed point theorem for four maps in partial b-metric-like spaces, SQU Journal for Science., 19(2),(2015), 55-61.

[12] Lakshmikantham, V. and Ciric, B. Lj., Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces, Nonlinear Analysis, 70 (2009), 4341-4349.

[13] Lakzian H. and Samet, B., Fixed points for ψ, φ - weakly contractive mappings in generalized metric spaces, Appl. Math. Lett., 25 (2012), 902 - 906.

[14] Matthews, S.G., Partial metric topology, in: Proc. of the 8 th Summer Conference on General Topology and Applications, Ann. New York Acad. Sci., 728(1994), 183-197.

[15] Moradi S. and Farajzadeh, A., On fixed point of ψ, φ - weakly and generalized ψ, φ - weak contraction mappings, Appl. Math. Lett., 25 (2012), 1257 - 1262.

[16] Musraleen, M., Mohouddine SA. and Agarwal, RP., Coupled fixed point theorems for α, ψ contractive type mappings in partially ordered metric spaces, Fixed Point Theory and Applications, 2012, 2012:228, 11 pages.

[17] Popescu, O., Fixed points for ψ, ϕ -weak contractions, Appl. Math. Lett., 24(2011), 1 - 4.

[18] Rao, K.P.R., Altun, I. and Hima Bindu, S., Common coupled fixed point theorems in generalized fuzzy metric spaces, Advances in fuzzy systems, 2011, Article ID 986748, (2011), 6 pages.

[19] Rao, K.P.R., Hima Bindu, S. and Mustaq Ali, Md., Coupled fixed point theorems in d- Complete topological spaces, J. Nonlinear Sci. Appl., 5 (2012), 186-194.

[20] Rao K.P.R. and .Kishore, G.N.V., A unique common fixed point theorem for four maps under ψ - ϕ contractive condition in partial metric spaces, Bull. Math. Anal. Appl., 3(2011), No.3, 56-63.

[21] Rao, K.P.R., Kishore G.N.V. and Van Luong, Nguyen, A Unique Common Coupled fixed point theorem for four maps under ψ- ϕ contractive condition in partial metric spaces, CUBO A Mathematical Journal, 14(03), (2012), 115-127.

[22] Rao, K.P.R., Kishore G.N.V. and Raju, V.C.C., A coupled fixed point theorem for two pair of w- compatible maps using altering distance function in partial metric spaces, Journal of Advanced Research in Pure Mathematics, 4(4), (2012), 96-114.

[23] Rao, K.P.R., Ranga Swamy P. and Rao, K.R.K., Common coupled fixed points for four maps using α -admissible functions in partial metric spaces, Jourrnal of the International Mathematical virtual Institute, 6,(2016), 103-118.

[24] Rao, K.P.R., Rao, K.R.K. and Sedghi, S., Common coupled fixed point theorem in fuzzy metric spaces, Gazi University Journal of Science, 27(2) (2014), 739-745.

[25] Rhoades, B.E., Some theorems on weakly contractive maps, Nonlinear Anal., 47, (2001),2683 - 2693.

[26] Seonghoon Cho, Fixed point theorems for generalized weakly contractive mappings in metric spaces with applications, Fixed Point Theory. Appl.,2018 (2018), Article No. 3, 18 pages.